A new mean-field-like renormalization group transformation

1994 J. Phys. A: Math. Gen. 27 1495

(http://iopscience.iop.org/0305-4470/27/5/015)

J. Phys. A: Math. Gen. 27 (1994) 1495–1502. Printed in the UK

A new mean-field-like renormalization group transformation

J Kamphorst Leal da Silva†, R Alves Nogueira†, J A Plascak†, and M Droz‡

† Departamento de Física, Instituto de Ciências Exatas, Universidade Federal de Minas Gerais, CP 702, 30161-970 Belo Horizonte, Brazil

‡ Department of Theoretical Physics, University of Geneva, 1211 Geneva 4, Switzerland

Received 27 September 1993, in final form 23 December 1993

Abstract. A new real space renormalization group transformation combining ideas from mean-field and finite-size scaling theories is presented. Application to the two-dimensional site directed percolation problem gives better values for the percolation threshold and critical exponent of longitudinal correlation length than those obtained previously with other real space renormalization group approaches. When applied to (one-, two- and three-dimensional) Ising models, the results are comparable to the ones from previous mean-field-like renormalization group transformations. This method can easily be applied to other systems having second-order phase transitions.

A large body of work has been devoted to the study of approximate real space renormalization group transformations which can be used in many different contexts.

Several of these approaches are based on the finite-size scaling hypothesis [1]; for example 'phenomenological' renormalization [2] which compares the correlation length for systems of different finite sizes and, more recently, 'finite-size scaling renormalization' [3] in which one studies quantities with zero anomalous dimension.

Another approach is the so-called mean-field renormalization group (MFRG), originally proposed by Indekeu *et al* [4]. It is mean-field-like in the sense that one computes the value of the order parameter for small clusters in the presence of a boundary field breaking the symmetry. Using renormalization group arguments, comparison of the order parameters obtained for clusters of different sizes allows a recursion relation for the couplings of the model to be extracted. However, consistency arguments make it necessary to consider bulk and surface ordering simultaneously [5]. Accordingly, three different exponents have to be extracted from the data obtained by comparing the results of three clusters of different size. Although this approach provides a simple method to find flow in the parameter space and to compute critical exponents, it suffers some drawbacks. The first is related to the fact that the scaling factor associated with the transformation is not always unambiguously defined. As a result, the extrapolation processes can be rather difficult. The second is that some spurious fixed points, without physical meaning, can show up.

In this paper we propose a new real space renormalization group transformation based uniquely on mean-field and finite-size scaling ideas and which is free of the above drawbacks. Although close in spirit to the MFRG, this approach does not refer to surface scaling and shows no ambiguity in the choice of the scaling factor.

Let us expose the general outline of the method in the case of a phase transition described by a scalar order parameter P_N. Let p be the control parameter and h, the conjugated external field. Let us consider a cluster formed of N sites. In zero external field, the only way to obtain a non-zero value for the order parameter is to add, at the boundaries of

0305-4470/94/051495+08$19.50

the cluster, a symmetry breaking field b; thus $P_N = f_N(p, h, b)$. In the extended mean-field procedure proposed by Bethe–Peirls [6], the breaking field is identified with the order parameter itself and then P_N is the solution of the implicit equation

$$P_N = f_N(p, h, P_N). \tag{1}$$

For a whole interval of p ('disordered phase'), the only solution of the above equation is $P_N = 0$. The threshold value p_N^c is a bifurcation point from which the equation has more than one solution. This defines the critical coupling p_N^c. The use of larger clusters improves the value of p_N^c, but the critical exponents $\alpha^*, \beta^*, \delta^*, \gamma^*, \ldots$ keep their classical (mean-field) values.

The corresponding equation of state takes its usual scaling form

$$h(P_N, \epsilon_N) = P_N^{\delta^*} \Phi_N[\epsilon_N / P_N^{1/\beta^*}] \tag{2}$$

where $\epsilon_N = p - p_N^c$, and β^* and δ^* are the usual order parameter exponents [7]. In the vicinity of the critical point and to first order in h, P_N can be written as a function of ϵ_N and h as

$$P_N(p, h) \approx a_N(p, p_N^c)\epsilon_N^{\beta^*} + h c_N(p, p_N^c)\epsilon_N^{-\gamma^*}. \tag{3}$$

The functions $a_N(p, p_N^c)$ and $c_N(p, p_N^c)$ are, in the limit $p = p_N^c$, the so-called anomaly coefficients introduced by Suzuki [8]. Close to criticality, these coefficients approach some constant values and, accordingly, one can view P_N as a function of ϵ_N and h. Thus, from equation (3) it follows that $P_N(\epsilon_N, h)$ has the following homogeneity property:

$$P_N(\lambda^{1/\beta^*}\epsilon_N, \lambda^{1+(\gamma^*/\beta^*)}h) = \lambda P(\epsilon_N, h). \tag{4}$$

Using a similar argument to that used by Indekeu *et al* [4], we interpret the results obtained for two different cluster sizes N and N' in terms of a scale transformation, namely

$$P_{N'}(p', h') = \lambda^x P_N(p, h) \tag{5}$$

where the scaling factor λ^x is unknown because the order parameters have been computed by using an approximation. However, the problem of the determination of this factor can be avoided by noticing the following fact: from (4), $\epsilon_N^{\beta^*}$ obeys, under rescaling, an equation similar to the one for P_N. Accordingly, it is reasonable to assume that

$$\epsilon_{N'}^{\prime\beta^*} = \lambda^x \epsilon_N^{\beta^*}. \tag{6}$$

The above relation is the key assumption of the method. This relation suggested by mean-field ideas is by no means exact, but should provide a very good approximation.

Let us first consider the case of zero external field. It follows from (3), (5) and (6) that

$$a_{N'}(p', p_{N'}^c) = a_N(p, p_N^c). \tag{7}$$

In terms of renormalization group transformation this equation can be interpreted as a recursion relation for the control parameter p

$$p' = \Theta_{N,N'}(p) \tag{8}$$

which defines a flow in the one-dimensional parameter space p. Among the several possible fixed points, one should correspond to the critical point $p^*_{N,N'}$ and the linearization of the recursion relations around this fixed point will give the anomalous dimension y_p, namely

$$\left.\frac{\mathrm{d}p'}{\mathrm{d}p}\right|_{p^*} = \left(\frac{N}{N'}\right)^{y_p/d} \tag{9}$$

where p^* and y_p depend on N and N'. In the presence of an external field, the associated anomalous dimension y_h is given as usual by

$$\frac{\mathrm{d}h'}{\mathrm{d}h} = (N/N')^{y_h/d}. \tag{10}$$

Using equations (3), (5) and (6) one finds

$$\left(\frac{N}{N'}\right)^{y_h/d} = \left(\frac{\epsilon'_{N'}}{\epsilon_N}\right)^{\beta^*+\gamma^*} \left.\frac{c_N(p, p^c_N)}{c_{N'}(p', p^c_{N'})}\right|_{p^*}. \tag{11}$$

We can see that, although in the present context any approximation defining the coefficients a_N and c_N is based on classical exponents β^* and γ^*, non-classical exponents can be extracted from the linearization of the recursion relation around the fixed point.

The larger the values of N and N' are, the better the approximation will be because more fluctuations will have been included in the mean-field calculation of each cluster. Moreover, the scaling assumption (6) will be more accurate as the scaling ratio N/N' approaches 1. Accordingly, one anticipates that the best values of the critical parameter p and the best value of the anomalous dimension y_p will be obtained in the limit $N \to \infty$ and $\rho = N/N' \to 1$. In this limit we expect the extraction of the fixed point to be a stable operation and that the sequence of $p^*_{N,N'}$ will converge to the exact value when the corresponding sequence of mean-field estimates p^c_N also converges (rather more slowly) to the exact value. Therefore, the clusters must be chosen in such a way that, by increasing their size, the mean-field critical parameters approach the true value.

As a first example, we shall study the problem of directed percolation. We restrict ourselves here to the two-dimensional site directed problem. Let us briefly remind ourselves of the model. The sites of a two-dimensional square lattice are occupied with independent probability p. Sites are related among themselves by directed bonds pointing towards a particular direction z (the vertical one in figure 1). As p increases, the cluster formed by the site orientationally connected to the origin becomes larger and larger. At the threshold value p_c, an infinite cluster spanning the whole system is formed. The order parameter is the probability $P(p)$ that the site at the origin is a source of an infinite cluster [9] and that p is the control parameter. The external field coupling to the order parameter corresponds to the probability that a site is coupled by a bond to an external 'ghost site' [10]. One characteristic feature of this problem is the presence of two different correlation lengths, parallel and perpendicular to the z direction. Accordingly, two different exponents $\nu_\parallel$ and $\nu_\perp$ have to be introduced. This anisotropy makes it difficult to devise a consistent real space renormalization group transformation [11, 12]. However, the directed percolation problem is particularly well suited to test the method because it is possible to study large clusters.

The mean-field part of the calculation follows the work of De'Bell [11]. The two smallest clusters ($N = 1$ and $N = 3$) are depicted in figure 1. Note that the empty sites—here called mean-field sites—are adjacent to the clusters. The effective field b is

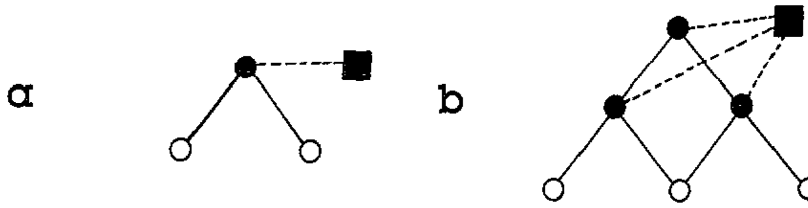

Figure 1. The asymmetric mean-field clusters with (*a*) $N = 1$ and (*b*) $N = 3$ sites. Empty circles represent the mean-field sites. The dashed lines are the bonds connecting the sites of the cluster to the ghost site (represented by a square).

the probability that a mean-field site will be connected to the infinite cluster. Let us now consider explicitly the evaluation of $P_N(p, h, b)$ for the cluster with $N = 1$ (see figure 1(*a*)). The bond connecting the origin to the ghost site is present with probability h. When this happens, one has a percolating configuration which is independent of the sites and mean-field sites. Then all configurations with the origin connected to the ghost site contribute to a term h in $P_1(p, h, b)$. When this bond is absent (with probability $1 - h$), we must consider configurations with at least one mean-field site connected to the percolation cluster. These configurations give us a term $(1-h)\{pb^2+2pb(1-b)\}$ in the percolation probability. Near criticality we have

$$P_1 = h(1 - 2pb) + p(-b^2 + 2b) \tag{12}$$

up to terms of order of hb and b^2. The mean-field approximation is obtained by setting $b = P_1$. For p greater than but near the percolation threshold we obtain

$$P_1(p, h) = a_1(p, p_1^c)\epsilon_1 + (h/\epsilon_1)c_1(p, p_1^c) \tag{13}$$

with $\epsilon_1 = p - p_1^c$, $p_1^c = 1/2$, $a_1(p, p_1^c) = 2/p$ and $c_1(p, p_1^c) = 1/2$. As expected, the critical exponents take their classical values $\beta^* = 1$ and $\gamma^* = 1$. A similar calculation can be done for the cluster with $N = 3$ sites (see figure 1(*b*)) and one gets

$$P_3(p, h) = a_3(p, p_3^c)\epsilon_3 + (h/\epsilon_3)c_3(p, p_3^c) \tag{14}$$

$$a_3(p, p_3^c) = \left[-p^2 + \frac{p}{(p_3^c)^2} + \frac{1}{p_3^c}\right][p^3 + 2p^2]^{-1} \tag{15}$$

$$c_3(p, p_3^c) = [1 + 2p]\left[-p^2 + \frac{p}{(p_3^c)^2} + \frac{1}{p_3^c}\right]^{-1} \tag{16}$$

where $p_3^c = 0.5374$. The recursion relation for p is obtained from equation (7), namely

$$\frac{p'}{2} = (p^3 + 2p^2)\left(-p^2 + \frac{p}{(p_3^c)^2} + \frac{1}{p_3^c}\right)^{-1}. \tag{17}$$

The critical fixed point of this equation is $p^* = 0.7031$, in good agreement with the best estimations (0.7055 [13] or 0.7065 [14]). The critical exponent $\nu_\parallel = 1/y_p$ obtained from equation (9) describes the divergence of $\xi_\parallel$ [15]. For the two smallest clusters we find

$y_p = 1.149$. This value is very far from the expected one ($y_p = 0.58$ [13]). Moreover, the exponent y_h evaluated from equation (10) is $y_h = 1.171$.

A computer code has been written to compute exactly $P_N(p, h)$ for arbitrary N. Data have been obtained for cluster size up to 105 sites. For each value of N, the mean-field critical threshold, the coupling at the critical fixed point and the exponents are computed. The results are quoted in table 1. Note that for all values of N, the corresponding values of p at the fixed points are close to the best known value. Finite-size scaling theory suggests extrapolating the data according to $\xi_\| \sim \sqrt{N}$. A rather good least-square fit of $\ln(p_c - p^*) \times y_p \ln\sqrt{N/N'}$ is obtained for all reasonable values of y_p with N' fixed. The second row of table 2 shows the variation of the central estimate of p_c with the number of data points (M) considered in the case $N' = 3$ and $N = 6\ldots105$. We used the value of y_p given by our approach ($1/y_p = 1.68$). This leads to the extrapolated value, $p_c = 0.7062 \pm 1$ (the error being in the last digit). Our estimate is slightly different from the best series estimate [13] and closer to the value of the recent evaluation by ben-Avraham [14]. It is worth mentioning that the mean-field values of the critical threshold (not shown here) also approach the expected value but with a slower convergence. The extrapolation procedure for the exponent y_p follows a similar line to the one used in [12] and [16]. More explicitly, one considers the scaling factor in the limit $\sqrt{N/N'} \to 1$. Then the sequence of estimates for y_p can be extrapolated by using the relation $y_p = y_p^{\text{true}} + \delta K$, where $\delta = \sqrt{N/N'} - 1$, K is a positive constant and y_p^{true} is the true (or extrapolated) value of y_p. The intercept of $y_p \times \delta$ in the asymptotic regime gives y_p^{true}. A similar analysis can be done for the exponent y_h. These different estimations are shown in the two last rows of table 2. It follows that the best extrapolations are, respectively, $y_p = 0.595 \pm 6$ and $y_h = 1.326 \pm 6$. The value of y_p is in good agreement with the best expected one ($y_p = 0.58$ [13]).

Table 1. The critical parameters for clusters with N and N' sites; p^* is the fixed point, y_p and y_h are the critical exponents.

N	N'	p^*	y_p	y_h
3	1	0.70308	1.1489	1.1706
6	3	0.69301	0.9214	1.2098
10	6	0.69752	0.8390	1.2283
15	10	0.69951	0.7923	1.2418
21	15	0.70077	0.7607	1.2520
28	21	0.70166	0.7377	1.2602
36	28	0.70233	0.7200	1.2671
45	36	0.70285	0.7060	1.2729
55	45	0.70326	0.6946	1.2781
66	55	0.70360	0.6850	1.2827
78	66	0.70388	0.6769	1.2868
91	78	0.70411	0.6698	1.2905
105	91	0.70431	0.6637	1.2940

Note that this is the first real space renormalization group transformation which produces a good value of y_p. On the other hand, the difficulties encountered in previous real space renormalization group transformations concerning the determination of the transverse correlation length exponent [11, 12, 15] are also present in this approach.

As a second example, we shall consider the nearest-neighbours ferromagnetic Ising model. The order parameter, control parameter and external field are, respectively, the spin magnetization, reduced coupling constant K and the external reduced magnetic field h.

Table 2. Estimates of the critical parameters for clusters up to 105 sites. The central estimate with the number M of considered points in the fit is shown. The critical threshold p_c was obtained in the limit $\sqrt{N/N'} \to \infty$ with the cluster of $N' = 3$ fixed. The estimates of the critical exponents y_p and y_h were obtained in the limit $\sqrt{N/N'} \to 1$.

M	$p_c(M)$	$y_p(M)$	$y_h(M)$
12	0.70635	0.614	1.305
11	0.70616	0.608	1.312
10	0.70611	0.603	1.316
9	0.70611	0.599	1.320
8	0.70611	0.597	1.324
7	0.70612	0.595	1.326
6	0.70613	0.593	1.329
5	0.70614	0.591	1.331
4	0.70615	0.589	1.333
series expansions [13]	0.70549	0.577	
transfer-matrix [14]	0.70652		

A first test of the validity of the method consists in studying the one-dimensional case. We shall not give here the details of this simple calculation. The main features are the following: first, for all cluster sizes, one finds only two fixed points, $K = 0$ and $K = \infty$, with the expected flow and at which the anomalous dimension y_h takes its exact value. Moreover, the anomalous dimension y_p converges towards its exact value when $N \to \infty$ and $N/N' \to 1$.

The two- and three-dimensional cases are more interesting. The considered clusters have, respectively, square and cubic shapes and for computational reasons the maximum number of spins in a cluster is 27.

For zero external field, and near criticality, the equation of state for the magnetization m_N of a N spins cluster reads:

$$m_N = f(K)m_N + g_N(K)m_N^3 + O(m_N^5). \tag{18}$$

Hence

$$m_N = \sqrt{(1 - f_N(K))/g_N(K)}. \tag{19}$$

The mean-field critical coupling K_N^c is the solution of $1 - f_N(K_N^c) = 0$. Thus one has

$$a_N(K, K_N^c) = \sqrt{\frac{1 - f_N(K)}{g_N(K)}}\sqrt{\frac{K}{K - K_N^c}}. \tag{20}$$

Now,

$$\epsilon_N = (K - K_N^c)/K.$$

For non-zero field, the coefficients $c_N(K, K_N^c)$ are obtained by computing the linear susceptibility.

The results obtained for the critical coupling K_c and the anomalous dimensions y_K and y_h are given in tables 3 and 4 for several cluster sizes N and N'. The extrapolation has been made following the usual finite-size scaling prescriptions.

For the two-dimensional case, the critical coupling constant, the thermal anomalous dimension y_K and the field anomalous dimension y_h differ from their exact values by 3,

Table 3. Two-dimensional Ising model. The critical parameters for clusters with N and N' sites; K^* is the fixed point and y_t and y_h are the critical exponents.

N	N'	K^*	y_t	y_h
4	1	0.39938	0.7527	1.0545
9	1	0.40622	0.7808	1.1116
16	1	0.41078	0.7972	1.1490
25	1	0.41408	0.8094	1.1762
Extrapolation		0.427		
4	1	0.39938	0.7527	1.0545
9	4	0.41330	0.8289	1.2127
16	9	0.42029	0.8662	1.2987
25	16	0.42441	0.8858	1.3551
Extrapolation			0.94	1.50
Exact results		0.441	1.0	1.875

Table 4. Same as table 3 for the three-dimensional Ising model.

N	N'	K^*	y_t	y_h
8	1	0.21864	0.9647	1.0686
27	1	0.21919	1.0148	1.1418
27	8	0.21967	1.0996	1.2694
Series expansions		0.222	1.59	2.48

6, and 20% respectively. The results obtained for the critical coupling and the thermal anomalous dimension are comparable to the values obtained by Indekeu *et al* [5] with three cluster fits. Our estimation of the field anomalous dimension y_h is, however, poorer.

For the three-dimensional case, the critical coupling constant, the thermal anomalous dimension y_K and the field anomalous dimension y_h differ from their exact values by 1.2, 30, and 50% respectively. Although the value of the critical coupling is quite good, the anomalous dimensions are poor. This is not surprising in view of the very small number of clusters considered.

In conclusion, we see that this new mean-field-like renormalization group transformation leads to a coherent description in two different situations: directed percolation and the Ising model. This new approach is particularly well suited to the problems in which large clusters can be treated in mean-field approximation. This is the case of directed percolation for which the results obtained are better than the ones obtained previously by other real space renormalization group transformations. For the Ising case, where large clusters cannot be analysed in a reasonable computing time, the results are comparable to the ones obtained by other related approaches. This renormalization procedure has the advantage of being simple and provides a systematic extrapolation scheme. It could be applied to many different problems exhibiting second-order phase transitions. However, one has to keep in mind that this renormalization group transformation, like all the other real space renormalization group transformations, contains some uncontrolled approximations.

Acknowledgments

We would like to thank Dr J G Moreira for useful discussions. JKLS thanks the University of Geneva and MD thanks the Universidade Federal de Minas Gerais for kind hospitality. This work was partially supported by Brazilian agencies—Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) and Fundação de Amparo a Pesquisa do Estado de Minas Gerais (Fapemig)—and by the Swiss National Science Foundation.

References

[1] Fisher M E 1971 *Proc. Int. Summer School Enrico Fermi, Course 51 Critical Phenomena, (Varenna, Italy)* ed M S Green (New York: Academic)
[2] Nightingale M P 1976 *Physica* A **83** 561
[3] de Oliveira P M C 1992 *Europhys. Lett.* **20** 621
[4] Indekeu J O, Maritan A and Stella A L 1982 *J. Phys. A: Math. Gen.* **15** L291
[5] Indekeu J O, Maritan A and Stella A L 1987 *Phys. Rev.* B **35** 305
[6] Bethe H A 1935 *Proc. R. Soc. London* A **150** 552
Pierls R 1936 *Proc. R. Soc. London* A **154** 207
[7] Stanley H E 1971 *Introduction to Phase Transitions and Critical Phenomena* (Oxford: Clarendon)
[8] Suzuki M 1986 *J. Phys. Soc. Japan* **55** 4205
[9] Kinzel W and Yeomans J M 1981 *J. Phys. A: Math. Gen.* **14** 1163
[10] Reynolds P J, Stanley H E and Klein W 1977 *J. Phys. A: Math. Gen.* **10** L203
[11] De'Bell K 1983 *J. Phys. A: Math. Gen.* **16** 1279
[12] das Neves A and Kamphorst Leal da Silva J 1991 *J. Phys. A: Math. Gen.* **24** 5363
[13] Essam J W, Guttmann A J and De'Bell K 1988 *J. Phys. A: Math. Gen.* **21** 3815
[14] ben-Avraham D, Bidaux R and Schulman L S 1991 *Phys. Rev.* A **43** 7093
[15] Redner S 1982 *Phys. Rev.* B **25** 3242
[16] Reynolds P J, Stanley H E and Klein W 1978 *J. Phys. A: Math. Gen.* **11** L199